

The C-finite Ansatz meets the Holonomic Ansatz

By Shalosh B. EKHAD and Doron ZEILBERGER

VERY IMPORTANT

As in all our joint papers, the main point is not the article, but the accompanying Maple package, `CfiniteIntegral.txt`, that may be downloaded, free of charge, from the web-page of this article

<http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/cfiniteI.html> ,

where the readers can also find sample input and output files, that they are welcome to extend using their own computers.

Preface

In a recent article, [Kim], symbolic summation, and quite a bit of human pre-processing, is used to evaluate certain integrals involving Chebyshev polynomials.

First, let's remark that the Chebyshev polynomials, like all *classical orthogonal polynomials*, belong to the **Holonomic Ansatz** ([Z1], beautifully, and very efficiently, implemented in [Kou]), and as such, *inter alia*, **every** identity in [Kim] (and in many other articles that are still published today) are *automatically provable*, and their *epistemological status* is the same as identities like $(a^3 - b^3) = (a - b)(a^2 + 2ab + b^2)$ or $134 \cdot 431 = 57754$. Could you imagine a paper published today (or even two thousands years ago), entitled "A new proof of the identity $134 \cdot 431 = 57754$ "?,

Introduction

But the Chebyshev polynomials are not 'just' *holonomic*, they belong to the more restricted class of *C-finite* polynomial sequences, and hence belong to the *C-finite ansatz* ([Z2]), and as such have nice closure properties. It turns out that one can *interface* the *C-finite ansatz* and the *holonomic ansatz*, and borrow from the latter the powerful, and not-as-well-known-as-it-should-be

"the (continuous) **Almkvist-Zeilberger algorithm**",

described in [AZ], and implemented in the Maple package

<http://www.math.rutgers.edu/~zeilberg/tokhniot/EKHAD.txt> .

We combined all the procedures from the above Maple package and the Maple package that accompanies [Z2], and created a new, self-contained, Maple package,

<http://www.math.rutgers.edu/~zeilberg/tokhniot/CfiniteIntegral.txt> ,

that can **automatically** prove *every* identity in [Kim], and many, **far deeper**, identities.

We should note that [Kim] also uses computer algebra methods, but those involving *summation*, and spends quite a lot of *human effort* to go from integration to summation. With the Almkvist-Zeilberger algorithm, one can proceed directly, as follows.

Using the Continuous Almkvist-Zeilberger algorithm in order to Evaluate (Symbolically!) Integrals of Powers of [in particular] Chebyshev Polynomials

Suppose that you want to study a sequence of the form

$$a(n) := \int_{\alpha}^{\beta} P_n(x) K(x) dx \quad ,$$

for some “nice” *kernel*, $K(x)$, and a sequence $P_n(x)$ of C -finite polynomials, i.e. given by a recurrence

$$P_n(x) = \sum_{i=1}^L p_i(x) P_{n-i}(x) \quad ,$$

for some positive integer, L , (the *order*) and polynomials $p_i(x)$ ($i = 1, \dots, L$), subject to *initial conditions*

$$P_0(x) = q_0(x), \dots, P_{L-1}(x) = q_{L-1}(x) \quad ,$$

for some polynomials $q_0(x), \dots, q_{L-1}(x)$.

What’s nice about the C -finite ansatz is that once you know that $\{P_n(x)\}$ is C -finite, the same is true for $\{P_n(x)^r\}$ for any positive integer r , and also the sequence $P_n(x)P_n^*(x)$ (where for any polynomial $a(x)$, $a^*(x) = x^d a(1/x)$ (where d is the degree of $a(x)$), is its “reverse”), and many other related sequences, and one can fully automatically (and very fast) find C -finite representations for them (see [Z2], and [KP] (a true **masterpiece!**)).

Once we have such a C -finite polynomial sequence, the *ordinary* generating function

$$R(x, t) := \sum_{n=0}^{\infty} P_n(x) t^n \quad ,$$

is a certain *rational function*, $R(x, t)$, of the variables x and t , that can be easily, and quickly, found automatically. Hence the (ordinary) generating function of the sequence $a(n)$, let’s call it $f(t)$,

$$f(t) := \sum_{n=0}^{\infty} a(n) t^n \quad ,$$

can be expressed as

$$f(t) = \int_{\alpha}^{\beta} R(x, t) K(x) dx \quad .$$

The continuous Almkvist-Zeilberger algorithm produces a *linear differential operator* with *polynomial* coefficients, $\mathcal{P}(t, \frac{d}{dt})$, and a *certificate* (a rational function times the integrand), let's call it $C(x, t)$, such that

$$\mathcal{P}(t, \frac{d}{dt}) [R(x, t)K(x)] = \frac{d}{dx}C(x, t) \quad .$$

Integrating with respect to x , from $x = \alpha$ to $x = \beta$, we get

$$\mathcal{P}(t, \frac{d}{dt}) \left[\int_{\alpha}^{\beta} R(x, t)K(x) dx \right] = \int_{\alpha}^{\beta} \left(\frac{d}{dx}C(x, t) \right) \quad .$$

Hence, by the *Fundamental Theorem of Calculus*, $f(t)$ satisfies an *inhomogeneous* ordinary differential equation with polynomial coefficients

$$\mathcal{P}(t, \frac{d}{dt})f(t) = C(\beta, t) - C(\alpha, t) \quad . \quad (DiffEq)$$

It is readily seen that the right-hand-side is a rational function in t . The differential equation (*DiffEq*) immediately implies a *linear (inhomogeneous) recurrence equation with polynomial coefficients* for the actual coefficients, namely, our original $a(n)$, from which, using standard methods, one can get a *homogeneous* linear recurrence equation with polynomial coefficients, that together with the *initial conditions*, that can be easily computed, constitutes a full **description** of the sequence $a(n)$, that enables us to compute the sequence to as-many-as-desired terms.

In fact, since we have the *theoretical guarantee* that such a description **exists** we can even skip the above steps, and resort to *pure guessing*!

As we have mentioned at the beginning, the above-mentioned web-page

<http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/cfiniteI.html> ,

contains numerous sample input and output files (including fully automatic proofs of the results of [Kim]), that readers can extend to their heart's content.

References

[AZ] Gert Almkvist and Doron Zeilberger, *The Method of Differentiating Under The Integral Sign*, J. Symbolic Computation **10**(1990), 571-591. Available from <http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/duis.html> .

[KP] Manuel Kauers and Peter Paule, *"The Concrete Tetrahedron"*, Springer, 2011.

[Kim] Seon-Hong Kim, *On some integrals involving Chebyshev polynomials*, Ramanujan Journal **38** (2015), 629-639.

[Kou] Christoph Koutschan, *Holonomic functions in Mathematica*, ACM Communications in Computer Algebra **47**(2013), 179-182. Available from <http://www.koutschan.de/publ/Koutschan13b/holofunc.pdf> .

[Z1] Doron Zeilberger, *A Holonomic Systems Approach To Special Functions*, J. Computational and Applied Math **32** (1990), 321-368. Available from <http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/holonomic.html> .

[Z2] Doron Zeilberger, *The C-finite Ansatz*, Ramanujan Journal **31** (2013), 23-32. Available from <http://www.math.rutgers.edu/~zeilberg/mamarim/mamarimhtml/cfinite.html> .

Doron Zeilberger, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.
zeilberg at math dot rutgers dot edu ; <http://www.math.rutgers.edu/~zeilberg/> .

Shalosh B. Ekhad, c/o D. Zeilberger, Department of Mathematics, Rutgers University (New Brunswick), Hill Center-Busch Campus, 110 Frelinghuysen Rd., Piscataway, NJ 08854-8019, USA.

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